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LETTER TO THE EDITOR

On the correspondence between the self-consistent 2D Poisson–Boltzmann structures and the sine-Gordon waves

N Martinov and N Vitanov

Faculty of Physics, University of Sofia, blvd. A. Ivanov 5, Sofia 1126, Bulgaria

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Abstract. By means of the transformation connecting the sine-Gordon equation and the two-dimensional Poisson–Boltzmann equation a correspondence between the self-consistent two-dimensional Poisson–Boltzmann structures and the solutions of the sine-Gordon equation representing standing waves has been established. In this way two new solutions of the sine-Gordon equation were obtained and studied and their application for description of waves into ferromagnetics was examined.

The Poisson–Boltzmann equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\sinh\psi \tag{1}$$

is obtained from the Poisson equation:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-\frac{\rho}{\varepsilon_0}. \tag{2}$$

In (2) ρ is the density of charged particles that can be expressed by means of the concentrations of positive- and negative-charged particles:

$$\rho=(n_+-n_+)e \tag{3}$$

If n_+ and n_- obey the Boltzmann statistic law:

$$n_+=n_0\exp(-eV/kT) \qquad n_-=n_0\exp(eV/kT) \tag{4}$$

then (1) can be obtained from (2) if:

$$\psi=eV/kT. \tag{5}$$

In this paper the two-dimensional Poisson–Boltzmann equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\sinh\psi(x,y) \tag{6}$$

and its solutions whose mathematical form is:

$$\psi(x,y)=4\tanh^{-1}\{\bar f(x)\bar g(y)\}$$

are discussed. If $\bar f(x)=Af(\alpha x;k_1)$ and $\bar g(y)=g(\beta y;k_2)$ where A,α,β are parameters, f and g are real Jacobi elliptic functions and k_1 and k_2 are their corresponding elliptic integral modules, then:

$$\psi(x,y)=4\tanh^{-1}\{Af(\alpha x;k_1)g(\beta y;k_2)\}. \tag{7}$$

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The solutions of the two-dimensional Poisson–Boltzmann equation describe the plane distribution of the particles in a two-component Coulomb gas. The solutions of type (7) reveal a tendency to self-organization, e.g. periodic plane distribution for a self-consistent two-component Coulomb gas.

Solutions of type (7) were studied (Martinov and Ouroushev 1986) and it was shown that the following five different types of solution exist:

$$\psi_1 = 4\tanh^{-1}\{A\,\mathrm{cn}(\alpha x; k_1)\,\mathrm{cn}(\beta y; k_2)\} \tag{8}$$

$$k_1^2 = \frac{A^2\{\alpha^2(a^2-1)-1\}}{\alpha^2(A^2-1)^2} \qquad k_2^2 = \frac{A^2\{\beta^2(A^2+1)-1\}}{\beta^2(a^2-1)} \qquad \alpha^2+\beta^2 = \frac{A^2+1}{A^2-1}$$

$$\psi_2 = 4\tanh^{-1}\{A\,\mathrm{dn}(\alpha x; k_1)\,\mathrm{sn}(\beta y; k_2)\} \tag{9}$$

$$k_1^2 = 1-\frac{\alpha^2(A^2-1)/A^2-1}{\alpha^2(A^2-1)} \qquad k_2^2 = \frac{A^2\{\beta^2(A^2-1)-1\}}{\beta^2(A^2+1)} \qquad \alpha^2 = A^2\beta^2$$

$$\psi_3 = 4\tanh^{-1}\left\{A\,\mathrm{dn}(\alpha x; k_1)\frac{\mathrm{sn}(\omega y; k_2)}{\mathrm{cn}(\beta y; k_2)}\right\} \tag{10}$$

$$k_1^2 = 1+\frac{\alpha^2(A^2+1)/A^2-1}{\alpha^2(A^2+1)} \qquad k_2^2 = 1-\frac{A^2\{1-\beta^2(A^2+1)\}}{\beta^2(A^2+1)} \qquad \alpha^2 = a^2\beta^2$$

$$\psi_4 = 4\tanh^{-1}\left\{A\,\mathrm{dn}(\alpha x; k_1)\frac{\mathrm{cn}(\beta y; k_2)}{\mathrm{sn}(\beta y; k_2)}\right\} \tag{11}$$

$$k_1^2 = 1-\frac{1-\alpha^2(a^2+1)/A^2}{\alpha^2(A^2+1)} \qquad k_2^2 = 1-\frac{1-\beta^2(A^2+1)/A^2}{\beta^2(A^2+1)} \qquad \alpha^2+\beta^2 = \frac{A^2}{A^2+1}$$

$$\psi_5 = 4\tanh^{-1}\left\{A\,\mathrm{dn}(\alpha x; k_1)\frac{1}{\mathrm{sn}(\beta y; k_2)}\right\} \tag{12}$$

$$k_1^2 = \frac{\alpha^2(A^2-1)^2+a^2}{\alpha^2A^2(A^2-1)} \qquad k_2^2 = \frac{\beta^2(A^2-1)+A^2}{\beta^2A^2(A^2-1)} \qquad \alpha^2-\beta^2 = \frac{A^2}{A^2-1}.$$

The functions sn, cn, dn are real Jacobi elliptic functions (Abramowitz and Stegun 1964). The solutions (8)–(12) are functions of two independent parameters, α and A. They describe two possible types of self-consistent two-dimensional structures (Martinov and Ouroushev 1986).

The sine-Gordon equation is as follows:

$$\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial t^2} = \sin\phi(x,t). \tag{13}$$

In this letter the solutions of this equation are discussed. Their mathematical form is:

$$\phi = 4\tan^{-1}\{\bar{A}f(\alpha x; k_1)g(\bar{\beta}t; k_2)\}. \tag{14}$$

$\bar{A}$, α and $\bar{\beta}$ are parameters, f and g are real Jacobi elliptic functions; k_1 and k_2 are their corresponding elliptic integral modules. There exist three solutions (Costabile *et al* 1978).

Plasma oscillations:

$$\phi_1 = 4\tan^{-1}\{\bar{A}\,\mathrm{cn}(\alpha x; k_1)\,\mathrm{cn}(\bar{\beta}t; k_2)\} \tag{15}$$

$$k_1^2 = \frac{\bar{A}^2\{\alpha^2(\bar{A}^2+1)+1\}}{\alpha^2(\bar{A}^2-1)^2} \qquad k_2^2 = \frac{\bar{A}^2\{\bar{\beta}^2(\bar{A}^2+1)-1\}}{\bar{\beta}^2(\bar{A}^2+1)} \qquad \bar{\beta}^2-\alpha^2 = \frac{\bar{A}^2+1}{\bar{A}^2+1}.$$

Breather oscillations:

$$\phi_2 = 4\tan^{-1}\{\bar{A}\,\mathrm{dn}(\alpha x; k_1)\,\mathrm{sn}(\bar{\beta}t; k_2)\} \tag{16}$$

$$k_1^2 = 1 - \frac{1-\alpha^2(\bar{A}^2+1)/\bar{A}^2}{\alpha^2(\bar{A}^2+1)} \qquad k_2^2 = \frac{\bar{A}^2\{1-\bar{\beta}^2(\bar{A}^2+1)\}}{\bar{\beta}^2(\bar{A}^2+1)} \qquad \alpha^2 = \bar{A}^2\bar{\beta}^2.$$

Fluxon oscillations:

$$\phi_3 = 4\tan^{-1}\left\{A\,\mathrm{dn}(\alpha x; k_1)\frac{\mathrm{sn}(\beta t; k_2)}{\mathrm{cn}(\beta t; k_2)}\right\} \tag{17}$$

$$k_1^2 = 1 - \frac{\alpha^2(\bar{A}^2-1)/\bar{A}^2-1}{\alpha^2(\bar{A}^2-1)} \qquad k_2^2 = 1 - \frac{\bar{A}^2\{\bar{\beta}^2(\bar{A}^2-1)-1\}}{\bar{\beta}^2(\bar{A}^2-1)} \qquad \alpha^2 = \bar{A}^2\bar{\beta}^2.$$

Note that a transformation exists connecting the representation of the sine-Gordon solutions (14) to those of the Poisson–Boltzmann solutions (7) via the transformation:

$$\begin{aligned} \psi &= \mathrm{i}\phi \qquad & \beta &= \mathrm{i}\bar{\beta} \\ y &= -\mathrm{i}t \qquad & A &= \mathrm{i}\bar{A} \end{aligned} \tag{18}$$

ϕ, $\bar{\beta}$ and $\bar{A}$ are real parameters.

Due to the transformation (18) there exists a correspondence between the two-dimensional self-consistent Poisson–Boltzmann structures and the objects which are described by the solutions of the sine-Gordon equation.

If (18) is imposed on the solutions (8)–(12) the result will be: the solutions ψ_1 (8) and ϕ_1 (15) correspond; the solutions ψ_2 (9) and ϕ_2 (16) correspond; and the solutions ψ_3 (10) and ϕ_3 (17) also correspond.

Two solutions of the 2D Poisson–Boltzmann equation remain. The solution ψ_4 (11) corresponds to the solution:

$$\phi_4 = 4\tan^{-1}\left\{\bar{A}\,\mathrm{dn}(\alpha x; k_1)\frac{\mathrm{cn}(\bar{\beta}t; k_2)}{\mathrm{sn}(\bar{\beta}t; k_2)}\right\} \tag{19}$$

$$k_1^2 = 1 - \frac{1+\alpha^2(1-\bar{A}^2)/\bar{A}^2}{\alpha^2(1-\bar{A}^2)} \qquad k_2^2 = 1 + \frac{1-\bar{\beta}^2(1-\bar{A}^2)/\bar{A}^2}{\bar{\beta}^2(1-\bar{A}^2)} \qquad \alpha^2 - \bar{\beta}^2 = \frac{\bar{A}^2}{\bar{A}^2+1}.$$

The solution ψ_5 (12) corresponds to the solution

$$\phi_5 = 4\tan^{-1}\left\{\bar{A}\,\mathrm{dn}(\alpha x; k_1)\frac{1}{\mathrm{sn}(\bar{\beta}t; k_2)}\right\} \tag{20}$$

$$k_1^2 = \frac{\alpha^2(\bar{A}^2+1)^2-\bar{A}^2}{\alpha^2\bar{A}^2(\bar{A}^2+1)} \qquad k_2^2 = \frac{\bar{A}^2-\bar{\beta}^2(\bar{A}^2+1)}{\bar{\beta}^2\bar{A}^2(\bar{A}^2+1)} \qquad \alpha^2 + \bar{\beta}^2 = \frac{\bar{A}^2}{\bar{A}^2+1}.$$

In this letter the solutions (19) and (20) are studied. It is known that the elliptic integral modules k_1 and k_2 of the Jacobi elliptic functions are limited bilaterally:

$$0 \leqslant k_1 \leqslant 1 \qquad 0 \leqslant k_2 \leqslant 1. \tag{21}$$

From (21) it follows that the parameters α and $\bar{\beta}$ in ϕ_4 are limited:

$$\alpha^2 \geqslant \frac{\bar{A}^2}{(\bar{A}^2-1)^2} \qquad \bar{\beta}^2 \geqslant \frac{\bar{A}^2}{(\bar{A}^2-1)^2}. \tag{22}$$

The Jacobi elliptic functions appearing in (19) are periodic so that

$$\begin{aligned} &\operatorname{dn}(\alpha x; k_1) \to T_x = \frac{2K(k_1)}{\alpha} \\ &\frac{\operatorname{sn}(\bar{\beta}t; k_2)}{\operatorname{cn}(\bar{\beta}t; k_2)} \to T_t = \frac{2K(k_2)}{\bar{\beta}} \end{aligned} \tag{23}$$

where $K(k_1)$ and $K(k_2)$ are the corresponding elliptic integrals. From (22) and (23) it can be seen that the periods T_x and T_t are limited:

$$\begin{aligned} &T_x \leqslant 2K(k_1)(\bar{A}^2 - 1)/\bar{A} \\ &T_t \leqslant 2K(k_2)(\bar{A}^2 - 1)/\bar{A}^2. \end{aligned} \tag{24}$$

The following special case for solution (19) is possible:

$$k_1 = (1 - 1/\bar{A}^4) \qquad k_2 = 0. \tag{25}$$

Then

$$\phi_4 = \phi_{4,1} = 4\tan^{-1}\left\{\bar{A}\operatorname{dn}\left[\frac{\bar{A}^2 x}{\bar{A}^2 - 1}; \left(\frac{1-1}{\bar{A}^4}\right)^{1/2}\right]\cot\left(\frac{\bar{A}^2 t}{\bar{A}^2 - 1}\right)\right\}. \tag{26}$$

This is a very interesting combination between a special function and an elementary function. Here the period T_t is minimal. If (21) and (20) are combined the result will be:

$$\frac{\bar{A}^2}{(\bar{A}^2+1)^2} \leqslant \alpha^2 \leqslant \frac{\bar{A}^2}{\bar{A}^2+1} \qquad \frac{\bar{A}^2}{(\bar{A}^2+1)^2} \leqslant \bar{\beta}^2 \leqslant \frac{\bar{A}^2}{\bar{A}^2+1} \tag{27}$$

$$\begin{aligned} &2K(k_1)(A^2+1)^{1/2}\bar{A} \leqslant T_x \leqslant 2K(k_1)(\bar{A}^2+1)/\bar{A} \\ &4K(k_2)(a^2+1)^{1/2}\bar{A} \leqslant T_t \leqslant 4K(k_2)(\bar{A}^2+1)/\bar{A}. \end{aligned} \tag{28}$$

There are two special cases for the solution (20):

$k_1 = 0$, $k_2 = 1/\bar{A}^2$. Then

$$\phi_5 = \phi_{5,1} = 4\tan^{-1}\left\{\frac{\bar{A}}{\operatorname{sn}(\bar{A}^2 t/(\bar{A}^2+1); 1/\bar{A}^2\}}\right\}. \tag{29}$$

$\phi_{5,1}$ is independent of the variable x.

$k_1 = (1 - 1/\bar{A}^4)^{1/2}$, $k_2 = 1$. Then

$$\phi_5 = \phi_{5,2} = 4\tan^{-1}\left\{\bar{A}^2\operatorname{dn}\left\{\frac{\bar{A}^2 x}{\bar{A}^2+1}; \left(\frac{1-1}{\bar{A}^4}\right)^{1/2}\right\}\operatorname{cotanh}\left(\frac{\bar{A}^2 t}{\bar{A}^2+1}\right)\right\} \tag{30}$$

There is the combination between a special and an elementary function again. But now the elementary function is not a trigonometric one. It is a hyperbolic function. The period $T_t = \infty$ i.e. $\phi_{5,2}$ is not a periodic function against the variable t.

The type of the new solutions is the following: the solution (19) belongs to fluxon oscillations and the solution (20) is similar to the breather oscillations although (20) there are certain differences between them.

The sine-Gordon equations have many applications in physics. In this letter only one application—the description of the movement of the vectors of the magnetization at the weakly excited states of exchanged ferromagnetic—is discussed. In the approximation of a non-dissipative medium these states are described by the Landau–Lifshitz equation.

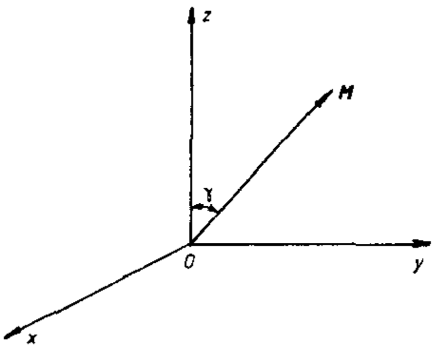

Figure 1. Situation of the vector of the magnetization in comparison with coordinate *Oxyz*. The assumption is that the vector *M* belongs to the plane *Oyz*.

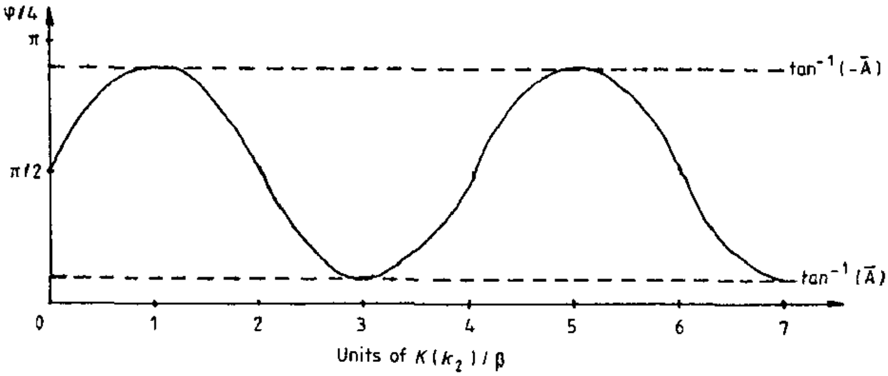

Figure 2. An oscillation of the vector of the magnetization described by means of the solution (20) of the sine-Gordon equation in the case where $x=0$.

On the assumption referring to a ferromagnetic which possesses an anisotropy whose type is an axis of easy magnetization the Landau-Lifshitz equation can be reduced to the sine-Gordon equation (Enz 1961). Then $\phi=2\gamma$, where γ is the angle, concluded from the vector of magnetization $\boldsymbol{M}$ with the axis Oz (figure 1).

Taking the solution (19)—if $t=0$, then $\gamma=\pi$—the vectors of magnetization regardless of their position conclude an angle π with the axis Oz. If t begins to increase, the angle γ begins to decrease and when $t=K(k_2)/\bar{\beta}$ then $\gamma=0$. Then γ continues to decrease. The result of this examination is that the vectors of magnetization in every point x rotate in the same direction—counter clockwise. The direction of rotation of the vectors of magnetization described in the solution (19) is opposite to the direction of rotation in the solution (17). This is the basic difference between the solutions (17) and (19).

The solution (20) describes the oscillations of the vectors of the magnetization around an average point. According to this (20) resembles (15) and (16). Along with this there are some features which belong to the solution (20). The solution (15) and (16) describe oscillations of the vector of the magnetization around $\phi=0$ or $\gamma=0$. The solution (20) describes the oscillations around $\phi=\pi/2$ or $\gamma=\varepsilon/4$. The next difference between the solutions (16) and (20) is that of the behaviour of the magnetization in time is imposed on by different functions, an sn-function for (16) and

1/sn-function for (20). The oscillations of the vector of the magnetization described from the solution (20) are presented in figure 2 in the case where $x = 0$.

References

Martinov N and Ouroushev D 1986 *J. Phys. A: Math. Gen.* **19** 2707
Costabile G, Parmentier R D, Savo B and McLaughlin D W 1978 *Appl. Phys. Lett.* **32** 587
Enz U 1961 *Helv. Phys. Acta* **31** 245
Abramowitz M and Stegun I 1970 *Handbook of Mathematical Functions* (New York: Dover)